

# EXPONENTIAL MIXING FOR SOME SPDES WITH LÉVY NOISE

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**ABSTRACT.** We show how gradient estimates for transition semigroups can be used to establish exponential mixing for a class of Markov processes in infinite dimensions. We concentrate on semilinear systems driven by cylindrical  $\alpha$ -stable noises introduced in [17],  $\alpha \in (0, 2)$ . We first prove that if the nonlinearity is bounded, then the system is ergodic and strong mixing. Then we show that the system is exponentially mixing provided that the nonlinearity, or its Lipschitz constant, are sufficiently small.

**Keywords:** stochastic PDEs driven by  $\alpha$ -stable noises, ergodicity, strong mixing, exponential mixing.

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## 1. INTRODUCTION

The paper is concerned with asymptotic properties of the stochastic evolution equation:

$$(1.1) \quad dX(t) = [AX(t) + F(X(t))]dt + dZ_t, \quad X_0 = x,$$

driven by  $\alpha$ -stable,  $\alpha \in (0, 2)$ , cylindrical process  $Z$ , introduced in [17]. The paper [17] investigated structural properties of  $X$  like strong Feller and irreducibility. Solutions turned out to be stochastically continuous but in general without càdlàg modifications, see [4]. The stochastic PDEs driven by Lévy noises have been intensively studied since some time, see e.g. [3], [1], [16], [13], [12], [17] and [20], and the book [15] for additional references. Even for equations like (1.1), with the additive noise, some basic questions are still open.

General results on existence of invariant measures for the linear equation

$$(1.2) \quad dX(t) = AX(t)dt + dZ_t, \quad X_0 = x,$$

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with  $A$  being the infinitesimal generator of a strongly continuous semigroup, were obtained in [6], and more recently in [18] and [10], with different assumptions on  $Z$ . The nonlinear case was investigated in [19], and by the dissipativity method in [15]; see also the recent paper [20].

In the present paper we are concerned with the exponential convergence to equilibrium, called exponential mixing, see e.g. [15, Proposition 16.5]. We assume that the nonlinearity  $F$  is Lipschitz continuous and bounded and find explicit conditions on  $F$  under which the exponential mixing takes place. The speed of the convergence to equilibrium can be deduced from the proofs. Our result seems to be new even in the well-studied case when  $Z$  is a cylindrical Wiener process, corresponding to  $\alpha = 2$ , requiring only obvious modifications of our proofs.

To prove the exponential mixing we establish first gradient estimates for the transition semigroup of the process  $X$  using the so called mild Kolmogorov equation. This version of the Kolmogorov equation was introduced in [5] and elaborated in [7] and [8]. With a similar aim gradient estimates were used in [20], in a different framework, with the assumption  $\alpha \in (1, 2)$  and without the exponential bounds. The ergodicity in [20] and [21] was obtained by the technique of interacting particle systems called finite speed of propagation of information. For the finite dimensional stochastic system driven by  $\alpha$ -stable processes, [20] obtained the strong mixing result for small  $F$ .

Our main results are formulated as Theorems 2.5 and 2.6. The first one establishes ergodicity without additional condition on  $F$ . Here it was enough to use the method sketched in [9]. Theorem 2.6 is on exponential mixing with additional restrictions on  $F$ . If the nonlinearity is only Lipschitz and bounded (and not small), it seems hard to obtain exponential mixing by the gradient estimates. We are now working on this general case trying a different method.

The organization of the paper is as follows. Section 2 introduces the notations and the main theorems. Section 3 proves the ergodicity and strong mixing for bounded nonlinearity. The next section is devoted to exponential gradient estimates. In the final section, we prove exponential mixing for small nonlinearities.

## 2. NOTATIONS AND MAIN RESULTS

We shall study our problem in a separable Hilbert space  $H$  with an orthonormal basis  $\{e_k\}_{k \geq 1}$ . Denote the inner product and norm of  $H$  by  $\langle \cdot, \cdot \rangle_H$  and  $|\cdot|_H$  respectively. Let  $C_b(H, H)$  be the Banach space of all bounded continuous functions  $f : H \rightarrow H$  with the supremum norm

$$\|f\|_0 := \sup_{x \in H} |f(x)|_H.$$

Similarly,  $C_b(H, \mathbb{R})$  denotes the Banach space of all bounded continuous functions  $f : H \rightarrow \mathbb{R}$  with the supremum norm  $\|f\|_0 := \sup_{x \in H} |f(x)|$ .

Given  $f \in C_b(H, \mathbb{R})$  and  $x \in H$ , for any  $h \in H$ , we write

$$(2.1) \quad D_h f(x) := \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon}$$

provided the above limit exists. If  $h \mapsto D_h f(x)$  is linear and  $|D_h f(x)| \leq C|h|_H$ ,  $h \in H$ , then there exists a unique element in  $H$ , denoted by  $Df(x)$ , such that  $D_h f(x) = \langle Df(x), h \rangle_H$ . Clearly,  $Df$  is a function from  $H$  to  $H$ . Denote by  $C_b^1(H, \mathbb{R})$  the set of all bounded differentiable functions  $f : H \rightarrow \mathbb{R}$  with norm

$$\|f\|_{C_b^1} := \sup_{x \in H} |f(x)| + \sup_{x \in H} |Df(x)|_H.$$

Let  $B_b(H, \mathbb{R})$  be the set of all bounded (Borel) measurable functions from  $H$  into  $\mathbb{R}$ .

Further, we denote by  $\mathcal{D}$  the set of all bounded continuous cylindrical functions  $f : H \rightarrow \mathbb{R}$ , i.e.,  $f$  is some bounded continuous function depending on finite number of coordinates of  $H$ . For any  $f \in \mathcal{D}$ , denote by  $\Lambda(f)$  the localization set of  $f$ , i.e.  $\Lambda(f)$  is the smallest set  $\Lambda \subset \mathbb{N}$  such that  $f \in C_b(\mathbb{R}^\Lambda, \mathbb{R})$ . Finally, define  $\mathcal{D}^1 := \mathcal{D} \cap C_b^1(H, \mathbb{R})$ .

Let  $z(t)$  be the normalized one dimensional symmetric  $\alpha$ -stable process,  $0 < \alpha < 2$ , with the following characteristic function

$$(2.2) \quad \mathbb{E}[e^{i\lambda z(t)}] = e^{-t|\lambda|^\alpha}, \quad \lambda \in \mathbb{R}.$$

The density of  $z(1)$  will be denoted by  $p_\alpha$ . The infinitesimal generator  $\partial_x^\alpha$  of the process is of the form

$$(2.3) \quad \partial_x^\alpha f(x) = \frac{1}{C_\alpha} \int_{\mathbb{R}} \frac{f(y+x) - f(x)}{|y|^{\alpha+1}} dy$$

with  $C_\alpha = -\int_{\mathbb{R}} (\cos y - 1) \frac{dy}{|y|^{1+\alpha}}$ , see [2].

We will consider equation (1.1) under the following assumptions:

**Assumption 2.1.**

- $A$  is a dissipative operator defined by

$$(2.4) \quad A = \sum_{k \geq 1} (-\gamma_k) e_k \otimes e_k.$$

$0 < \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_k \leq \dots$  and  $\gamma_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

- $F : H \rightarrow H$  is Lipschitz and bounded.
- $Z = (Z_t)$  is a cylindrical  $\alpha$ -stable process with  $Z_t = \sum_{k \geq 1} \beta_k z_k(t) e_k$  where  $\{z_k(t)\}_{k \geq 1}$  is a sequence of i.i.d. symmetric  $\alpha$ -stable processes (defined on some fixed stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  with  $0 < \alpha < 2$  and  $\beta_k > 0$ ,  $k \geq 1$ , and

$$(2.5) \quad \sum_{k \geq 1} \frac{\beta_k^\alpha}{\gamma_k} < +\infty.$$

- There exists  $\sigma \in (0, 1)$  such that

$$(2.6) \quad \sup_{k \geq 1} \frac{\gamma_k^{\frac{1}{\alpha} - \sigma}}{\beta_k} < +\infty.$$

*Remark 2.2.* Note that  $A$  generates a strongly continuous semigroup of compact contractions  $(e^{tA})$  on  $H$ . Moreover,  $e^{tA}e_k = e^{-t\gamma_k}e_k$ ,  $k \geq 1$ .

The first three conditions imply the existence of the unique mild solution  $X = (X(t, x))$  to (1.1), see [17]. This is a predictable  $H$ -valued stochastic process, depending on  $x \in H$ , such that, for any  $t \geq 0$ ,  $x \in H$ , it holds ( $\mathbb{P}$ -a.s.):

$$(2.7) \quad X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s, x))ds + Z_A(t), \quad \text{with } Z_A(t) = \int_0^t e^{(t-s)A}dZ_s.$$

Condition (2.6) is used in [17, Section 5] to get the strong Feller property for the Markov semigroup  $P_t$  associated with the solution  $X$  of (1.1):

$$P_t f(x) = \mathbb{E}[f(X(t, x))], \quad f \in B_b(H, \mathbb{R}), \quad x \in H, \quad t \geq 0.$$

For further use we need an equivalent formulation of (2.6).

**Lemma 2.3.** *For arbitrary  $\sigma > 0$  the following two conditions are equivalent:*

$$(2.8) \quad B = \sup_{k \geq 1} \frac{\gamma_k^{\frac{1}{\alpha} - \sigma}}{\beta_k} < +\infty,$$

$$(2.9) \quad k_t = \sup_{k \geq 1} \frac{e^{-\gamma_k t} \gamma_k^{1/\alpha}}{\beta_k} \leq \frac{\hat{c} e^{-\gamma_1 t/2}}{t^\sigma}, \quad t > 0,$$

where  $\hat{c} = B 2^{(2/\alpha) - \sigma} \frac{\sigma^\sigma}{e^\sigma}$ .

*Proof.* To establish the equivalence one can argue as in the beginning of [17, Section 5]. We only note the following estimate, for any  $n \geq 1$ ,

$$\frac{e^{-\gamma_n t} \gamma_n^{1/\alpha}}{\beta_n} \leq 2^{\frac{1}{\alpha}} \exp(-\gamma_1 t/2) \frac{e^{-\frac{\gamma_n}{2} t} (\gamma_n/2)^{1/\alpha}}{\beta_n}.$$

□

According to [17], we have the following result for the system (1.1).

**Theorem 2.4.** *Under the Assumption 2.1 there exists a unique mild solution  $X(t, x)$  for (1.1). Moreover its associated transition semigroup  $P_t$  is strong Feller and irreducible.*

This paper aims to study the long time behaviour of the system (1.1). The two main theorems are as follows.

**Theorem 2.5.** *Under Assumption 2.1 there exists a unique invariant measure  $\mu$  for the system (1.1). The measure  $\mu$  is strong mixing, i.e.*

$$\lim_{t \rightarrow \infty} P_t f(x) = \mu(f),$$

for all  $f \in B_b(H, \mathbb{R})$  and  $x \in H$ .

In the formulation of the next theorem  $p_\alpha$  stands for the density of  $z(1)$  and the constant  $\hat{c}$  was introduced in Lemma 2.3.

**Theorem 2.6.** *Assume that Assumption 2.1 holds and that one of the following two conditions holds:*

(i)  $L_F < \gamma_1$ , where  $L_F$  is the best Lipschitz constant of  $F$ ;

(ii)  $\|F\|_0 < \frac{C_0}{\Gamma(1-\sigma)} \left(\frac{\gamma_1}{2}\right)^{1-\sigma}$ , where  $C_0 = \hat{c} \int_{\mathbb{R}} \frac{(p'_\alpha(z))^2}{p_\alpha(z)} dz$ .

Then the system (1.1) is exponentially mixing. More precisely there exist constants  $C = C(|x|_H, \alpha, (\beta_n), (\gamma_n), F) > 0$  and  $c = c(\alpha, (\beta_n), (\gamma_n), F) > 0$  such that

$$(2.10) \quad |P_t f(x) - \mu(f)| \leq C e^{-ct} \|f\|_{C_b^1},$$

for all  $f \in C_b^1(H, \mathbb{R})$  and  $x \in H$ .

We do not give explicit formulas for the constants  $C$  and  $c$  but they can be obtained by a careful examination of the proofs.

Let us give some examples which the two main theorems can be applied to.

**Example 2.7.** Consider the following stochastic semilinear equation on  $D = [0, \pi]^d$  with  $d \geq 1$  with Dirichlet boundary conditions

$$(2.11) \quad \begin{cases} dX(t, \xi) = [\Delta X(t, \xi) + F(X(t, \xi))]dt + dZ_t(\xi), \\ X(0, \xi) = x(\xi), \\ X(t, \xi) = 0, \quad \xi \in \partial D, \end{cases}$$

where  $Z_t$  and  $F$  are both specified below. It is clear that  $\Delta$  with Dirichlet boundary condition has the following eigenfunctions

$$e_k(\xi) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \sin(k_1 \xi_1) \cdots \sin(k_d \xi_d), \quad k \in \mathbb{N}^d, \xi \in D.$$

It is easy to see that  $\Delta e_k = -|k|^2 e_k$ , i.e.  $\gamma_k = |k|^2 = k_1^2 + \dots + k_d^2$ , for all  $k \in \mathbb{N}^d$ . We study the dynamics (2.11) in the Hilbert space  $H = L^2(D)$  with orthonormal basis  $\{e_k\}_{k \in \mathbb{N}^d}$ .  $Z = (Z_t)$  is some cylindrical  $\alpha$ -stable noises which, under the basis  $\{e_k\}_k$ , is defined by

$$Z_t = \sum_{k \in \mathbb{N}^d} |k|^\beta z_k(t) e_k$$

where  $\{z_k(t)\}_k$  are i.i.d. symmetric  $\alpha$ -stable processes with  $\alpha \in (0, 2)$  and  $\beta$  a real number. Note that  $\sum_{k \in \mathbb{N}^d} \frac{|k|^{\beta\alpha}}{|k|^2} < \infty$  if and only if  $2 > d + \alpha\beta$ .

From Theorems 2.5 and 2.6, we have

- If  $F$  is a bounded Lipschitz function and

$$2 > d + \alpha\beta, \quad \frac{1}{\alpha} - \frac{\beta}{2} < 1,$$

or equivalently,

$$\frac{d}{\alpha} < \frac{2}{\alpha} - \beta < 2,$$

then the system (2.11) is strongly mixing.

- If in addition  $\|F\|_0$  is sufficiently small then the system (2.11) is exponentially mixing.

### 3. PROOF OF THEOREM 2.5

According to Sections 5.2 and 5.3 of [17], the system (1.1) is irreducible and strong Feller. By Doob's theorem (see [8, Theorem 4.2.1]), to prove Theorem 2.5, one only needs to show the existence of invariant measures. To this purpose it is enough to establish the tightness of  $\{\mathcal{L}(X(t, x))\}_{t \geq 1}$  for some  $x \in H$  (here  $\mathcal{L}(X(t, x))$  denotes the law of the random variable  $X(t, x)$ , see (2.7)). For this we basically follow [9].

Fix any  $x \in H$  and set  $X(t, x) = X(t)$ . First note that for any fixed  $t > 0$ ,  $e^{At}$  is a contraction and a compact operator. Contraction property is clear by the assumption of  $A$ . The compactness is an easy corollary of the fact that the eigenvalues  $-\gamma_k$  tend to  $-\infty$ .

According to [17, Proposition 4.2], for any (small)  $\varepsilon > 0$ , there exists some  $M = M_\varepsilon > 0$  such that

$$\mathbb{P}\{|Z_A(t)|_H \leq M\} \geq 1 - \varepsilon$$

uniformly for all  $t \geq 0$ . By the assumptions on  $F$  and  $A$ , one clearly has  $|\int_0^t e^{A(t-s)} F(X(s)) ds|_H \leq C$  uniformly for  $t \geq 0$  and  $\omega \in \Omega$ . Hence,

$$(3.1) \quad \mathbb{P}\{|X(t)|_H \leq |x| + C + M\} \geq \mathbb{P}\{|Z_A(t)|_H \leq M\} \geq 1 - \varepsilon.$$

Let us rewrite (2.7) as

$$X(t) = e^A X(t-1) + \int_{t-1}^t e^{A(t-s)} F(X(s)) ds + \int_{t-1}^t e^{A(t-s)} dZ_s,$$

thanks to the compactness of  $e^A$  and (3.1), the family  $\mathcal{L}(\{e^A X(t-1)\})_{t \geq 1}$  is tight. The integrals  $\int_{t-1}^t e^{A(t-s)} dZ_s$  have the same law as  $Z_A(1)$  for all  $t \geq 1$ , and thus their laws are of course tight.

To complete the proof of tightness of  $\{\mathcal{L}(X(t))\}_{t \geq 1}$ , it is enough to show that the values of the integrals  $\int_{t-1}^t e^{A(t-s)} F(X(s)) ds$  are contained in a compact set, for any  $t \geq 1$ . To this purpose note that the operator  $\mathcal{R}$  from  $L^2(0, 1; H)$  into  $H$

$$\mathcal{R}\phi = \int_0^1 e^{As} \phi(s) ds, \quad \phi \in L^2(0, 1; H),$$

is compact (this follows from the compactness of the operators  $e^{At}$ ,  $t > 0$ ). Using the compactness of  $\mathcal{R}$  and the fact that the transformation  $F$  is bounded, we get that the integrals are contained in a fixed compact set. The proof is complete.

#### 4. EXPONENTIAL GRADIENT ESTIMATES

To establish Theorem 2.6 we first derive exponential gradient estimates for the Galerkin approximation of the equation (1.1).

**4.1. Galerkin approximation.** Let  $\{e_k\}_{k \geq 1}$  be the orthonormal basis associated with the operator  $A$  and write  $\Gamma_N := \{1, \dots, N\}$

For any  $x \in H$  and any integer  $N > 0$  we have the following approximation of Eq. (1.1):

$$(4.1) \quad \begin{cases} dX_k^N(t) = [-\gamma_k X_k^N(t) + F_k^N(X^N(t))]dt + \beta_k dz_k(t), \\ X_k^N(0) = x_k, \end{cases}$$

for all  $k \in \Gamma_N$ , where  $x^N = (x_k)_{k \in \Gamma_N}$ ,  $x_k = \langle x, e_k \rangle_H$ , and  $F_k^N(x^N) = \langle F(x^N, 0), e_k \rangle_H$ . Eq. (4.1) can be written in the following vector form

$$(4.2) \quad \begin{cases} dX^N(t) = [AX^N(t) + F^N(X^N(t))]dt + dZ_t^N, \\ X^N(0) = x^N, \end{cases}$$

where  $X^N(t) = (X_k^N(t))_{k \in \Gamma_N}$  and  $Z_t^N = (\beta_k z_k(t))_{k \in \Gamma_N}$ . The infinitesimal generator of (4.2) is

$$(4.3) \quad \begin{aligned} \mathcal{L}_N &= \sum_{k \in \Gamma_N} \beta_k^\alpha \partial_k^\alpha + \sum_{k \in \Gamma_N} [-\gamma_k x_k + F_k^N(x^N)] \partial_k \\ &= \sum_{k \in \Gamma_N} [\beta_k^\alpha \partial_k^\alpha - \gamma_k x_k \partial_k] + \sum_{k \in \Gamma_N} F_k^N(x^N) \partial_k, \end{aligned}$$

where  $\partial_k = \partial_{x_k}$  and  $\partial_k^\alpha = \partial_{x_k}^\alpha$ . In the sequel we will identify  $x^N \in \mathbb{R}^N$  with

$$x^N = \sum_{k=1}^N x_k e_k \in H.$$

Consider the Kolmogorov equation of the Galerkin approximation

$$(4.4) \quad \begin{cases} \partial_t u^N(t) = \mathcal{L}_N u^N(t), \\ u^N(0) = f, \end{cases}$$

where  $f \in \mathcal{D}^1$  with  $\Lambda(f) \subset \Gamma_N$ . According to Section 5.3 of [17], Eq. (4.4) has a mild solution  $P_t^N f$  which satisfies

$$(4.5) \quad P_t^N f(x^N) = S_t^N f(x^N) + \int_0^t S_{t-s}^N [\langle F^N, DP_s^N f \rangle_H](x^N) ds,$$

where  $S_t^N$  is the Ornstein Uhlenbeck transition semigroup generated by the operator  $\sum_{k \in \Gamma_N} [\beta_k^\alpha \partial_k^\alpha - \gamma_k x_k \partial_k]$ . Moreover, we also have

$$P_t^N f(x^N) = \mathbb{E}[f(X^N(t, x^N))].$$

From the second step of the proof of [17, Theorem 5.7], we have

$$(4.6) \quad \lim_{N \rightarrow \infty} P_t^N f(x^N) = P_t f(x),$$

in particular, for any  $f \in \mathcal{D}^1$  with  $\Lambda(f) \subset \Gamma_N$ ,  $t \geq 0$ ,  $x \in H$ , where  $P_t$  is the transition semigroup defined in Theorem 2.4.

**4.2. Estimates and their proofs.** The gradient estimates which are established here are of two different types. The first one is straightforward, and although formulated for (4.1), is true in a much more general situation. Moreover, it is stated in terms of the best Lipschitz constant of the nonlinearity  $F$ . The second one is true for the specific systems considered in the paper, requires more subtle considerations and is stated in terms of the supremum of  $\{|F(x)|; x \in H\}$ .

**Proposition 4.1.** *Let  $P_t^N$  be the transition semigroup corresponding to the solution of the equation (4.1). Then*

$$(4.7) \quad \|DP_t^N f\|_0 \leq e^{-(\gamma_1 - L_F)t} \|Df\|_0, \quad f \in \mathcal{D}^1 \text{ with } \Lambda(f) \subset \Gamma_N, \quad t \geq 0.$$

*Proof.* Denoting by  $X^N(t, x^N)$  the solution of Eq. (4.1) with initial data  $x^N$ , one has

$$|X^N(t, x^N) - X^N(t, y^N)|_H \leq e^{-(\gamma_1 - L_F)t} |x^N - y^N|_H, \quad t \geq 0, \quad x, y \in H,$$

which implies  $\lim_{\varepsilon \rightarrow 0} \sup \left| \frac{X^N(t, x^N + \varepsilon h^N) - X^N(t, x^N)}{\varepsilon} \right| \leq e^{-(\gamma_1 - L_F)t} |h^N|$ ,  $h \in H$ . Hence, by the dominated convergence theorem one has

$$\begin{aligned} |D_h P_t^N f(x^N)| &\leq \mathbb{E} \left[ |Df(X^N(t))|_H \limsup_{\varepsilon \rightarrow 0} \left| \frac{X^N(t, x^N + \varepsilon h^N) - X^N(t, x^N)}{\varepsilon} \right| \right] \\ &\leq e^{-(\gamma_1 - L_F)t} \|Df\|_0 |h|_H. \end{aligned}$$

□

**Proposition 4.2.** *Let  $P_t^N$  be the transition semigroup corresponding to the solution of the equation (4.1). There exists a positive constant  $C$ , depending on  $\alpha, (\beta_n), (\gamma_n), \sigma$ , such that*

$$\|DP_t^N f\|_0 \leq C e^{-\omega t} \|Df\|_0, \quad f \in \mathcal{D}^1, \text{ with } \Lambda(f) \subset \Gamma_N, \quad t \geq 0,$$

where

$$\omega = \frac{\gamma_1}{2} - (C_0 \|F\|_0 \Gamma(1 - \sigma))^{\frac{1}{1-\sigma}},$$

and the constant  $C_0$  was introduced in Theorem 2.6.

To prove Proposition 4.2, we need the following lemma.



**Lemma 4.3.** *For any  $f \in \mathcal{D}^1$ , with  $\Lambda(f) \subset \Gamma_N$ , one has:*

$$(i) |DS_t^N f(x^N)|_H \leq e^{-\gamma t} \|Df\|_0,$$

$$(ii) |DS_t^N f(x^N)|_H \leq \frac{C_0 e^{-\gamma t/2}}{t^\sigma} \|f\|_0,$$

for all  $x \in H$ ,  $N \geq 1$ ,  $t > 0$  (where  $C_0 = \hat{c} \int_{\mathbb{R}} \frac{(p'_\alpha(z))^2}{p_\alpha(z)} dz$ , see Theorem 2.6).

*Proof.* Note that, for any vector  $h \in \sum_{k=1}^N h_k e_k$ ,  $x \in H$ , we have

$$\langle DS_t^N f(x^N), h \rangle = \langle DS_t f(x^N), h \rangle,$$

where  $S_t$  is the Ornstein-Uhlenbeck semigroup acting on  $C_b(H, \mathbb{R})$  associated to  $X(t, x)$  when  $F = 0$ . We know that

$$(4.8) \quad S_t f(x) = \int_H f(e^{tA}x + y) \mu_t^0(dy),$$

where  $\mu_t^0$  is the law of the random variable  $Z_A(t) = \int_0^t e^{(t-s)A} dZ_s$ . By differentiating under the integral sign in (4.8) we immediately get the first assertion (remark that  $\|e^{tA}\| \leq e^{-\gamma t}$ ,  $t \geq 0$ , where  $\|e^{tA}\|$  denotes the operator norm of  $e^{tA}$ ).

As for the second assertion, recall the gradient estimate from [17, Theorem 4.14]

$$|DS_t f(x)|_H \leq 8c_\alpha \left( \sup_{k \geq 1} \frac{e^{-\gamma_k t} \gamma_k^{1/\alpha}}{\beta_k} \right) \|f\|_0, \quad f \in C_b(H, \mathbb{R}), \quad t > 0,$$

where

$$c_\alpha = \frac{1}{8} \int_{\mathbb{R}} \frac{(p'_\alpha(z))^2}{p_\alpha(z)} dz.$$

According to (2.9) we have

$$(4.9) \quad |DS_t f(x)|_H \leq 8c_\alpha \frac{\hat{c} e^{-\gamma t/2}}{t^\sigma} \|f\|_0, \quad t > 0,$$

and the assertion follows.  $\square$

*Proof of Proposition 4.2.* By (4.5) and Lemma 4.3, we have, for any  $N \in \mathbb{N}$ ,

$$|DP_t^N f(x^N)| \leq e^{-\frac{\gamma}{2}t} \|Df\|_0 + \int_0^t \frac{C_0 e^{-\frac{\gamma}{2}(t-s)}}{(t-s)^\sigma} \|\langle F^N, DP_s^N f \rangle_H\|_0 ds,$$

where  $C_0$  is defined in Lemma 4.3; therefore,

$$\|DP_t^N f\|_0 \leq e^{-\frac{\gamma}{2}t} \|Df\|_0 + \int_0^t \frac{C_0 e^{-\frac{\gamma}{2}(t-s)}}{(t-s)^\sigma} \|F\|_0 \|DP_s^N f\|_0 ds,$$

Writing  $v^N(t) = e^{\frac{\gamma}{2}t} \|DP_t^N f\|_0$ , we have from the above inequality

$$v^N(t) \leq \|Df\|_0 + \int_0^t \frac{C_0 \|F\|_0}{(t-s)^\sigma} v^N(s) ds.$$

Now we use the following Henry's estimate (see [11]): let  $a \geq 0$ ,  $b \geq 0$  and  $\beta > 0$  and consider a non-negative locally integrable function  $u$  on  $[0, T)$  such that

$$u(t) \leq a + b \int_0^t (t-s)^{\beta-1} u(s) ds, \quad t \in [0, T),$$

then we have  $u(t) \leq a G_\beta(\theta t)$ ,  $t \in [0, T)$ , where

$$\theta = (b\Gamma(\beta))^{1/\beta}, \quad G_\beta(z) = \sum_{n \geq 0} \frac{z^{n\beta}}{\Gamma(n\beta + 1)}, \quad z \geq 0$$

(note that  $G_\beta(z) \sim \frac{1}{\beta} e^z$  as  $z \rightarrow +\infty$ ).

In our case  $u(t) = v^N(t)$ ,  $a = \|Df\|_0$ ,  $b = C_0\|F\|_0$  and  $\beta = 1 - \sigma$ . Thus for a constant  $C_1$ , depending on  $\sigma$ , and all positive  $t$  we get

$$v^N(t) \leq \frac{C_1\|Df\|_0}{1-\sigma} \exp \left\{ (C_0\|F\|_0\Gamma(1-\sigma))^{\frac{1}{1-\sigma}} t \right\}$$

Therefore

$$\|DP_t^N f\|_0 \leq \frac{C_1\|Df\|_0}{1-\sigma} \exp \left\{ (C_0\|F\|_0\Gamma(1-\sigma))^{\frac{1}{1-\sigma}} t - \frac{\gamma_1}{2} t \right\}$$

and we get the assertion.  $\square$

## 5. PROOF OF THEOREM 2.6

From Theorem 2.5, the system (1.1) is ergodic and has a unique invariant measure  $\mu$ .

Note that it is enough to prove (2.10) for any  $f \in \mathcal{D}^1$ . Indeed then, approximating any function  $f \in C_b^1(H)$  by a sequence  $(f_n) \subset \mathcal{D}^1$  such that  $f_n \rightarrow f$  and  $Df_n \rightarrow Df$  pointwise with  $\sup_{n \geq 1} \|f_n\|_{C_b^1} < \infty$ , we get easily the complete assertion. Let us fix  $f \in \mathcal{D}^1$  and suppose that  $\Lambda(f) \subset \Gamma_N$ .

The crucial point of the proof is to apply the gradient estimate from Proposition 4.2, in the spirit of [15, Proposition 16.4].

Concerning the case (i), by using (4.7), we can show the exponential mixing by a similar argument as for the case (ii). So from now on we shall concentrate on the proof of the case (ii).

First let us suppose that  $\alpha \in (1, 2)$ . Let

$$Z_A^N(t) = \int_0^t e^{A(t-s)} dZ_s^N$$

and write  $Y^N(t) = X^N(t, x^N) - Z_A^N(t)$  so that

$$Y^N(t) = e^{At} x^N + \int_0^t e^{A(t-s)} F^N(X^N(s, x^N)) ds.$$

For any  $t_2 > t_1 > 0$ , writing  $s = t_2 - t_1$ , by the gradient estimates of Proposition 4.2, we have, for any  $x \in H$ ,

$$\begin{aligned}
(5.1) \quad |P_{t_2}^N f(x^N) - P_{t_1}^N f(x^N)| &= |\mathbb{E}[P_{t_1}^N f(X^N(s, x^N)) - P_{t_1}^N f(x^N)]| \\
&\leq \|DP_{t_1}^N f\|_0 \mathbb{E}|X^N(s, x^N) - x^N|_H \\
&\leq Ce^{-\omega t_1} \|Df\|_0 \mathbb{E}|X^N(s, x^N) - x^N|_H,
\end{aligned}$$

where  $C, \omega > 0$  are given in Proposition 4.2. Now note that

$$|X^N(t, x^N) - x^N|_H \leq |e^{At}x^N - x^N|_H + \left| \int_0^t e^{A(t-s)} F^N(X^N(s, x^N)) ds \right|_H + |Z_A(t)|_H.$$

We have

$$\left| \int_0^t e^{A(t-s)} F^N(X^N(s, x^N)) ds \right|_H \leq \|F\|_0 \int_0^\infty e^{-\gamma_1 s} ds \leq \frac{\|F\|_0}{\gamma_1},$$

for any  $\omega \in \Omega$ ,  $x \in H$  and  $t \geq 0$ . Concerning  $|Z_A(t)|_H$  remark that since  $\alpha > 1$  (see [17, Theorem 4.4])

$$\mathbb{E}|Z_A(t)|_H \leq \tilde{c}_1 \left( \sum_{n \geq 1} |\beta_n|^\alpha \frac{(1 - e^{-\alpha \gamma_n t})}{\alpha \gamma_n} \right)^{1/\alpha} \leq \tilde{c}_1 \left( \sum_{n \geq 1} |\beta_n|^\alpha \frac{1}{\alpha \gamma_n} \right)^{1/\alpha} < +\infty, \quad t \geq 0.$$

It follows that

$$\mathbb{E}|X^N(s, x^N) - x^N|_H \leq C_2(1 + |x^N|_H),$$

where  $C_2$  does not depend on  $N$ ,  $s \geq 0$  and  $x \in H$ .

By (5.1) we get

$$|P_{t_2}^N f(x^N) - P_{t_1}^N f(x^N)| \leq C_3 e^{-\omega t_1} \|Df\|_0 (1 + |x^N|_H).$$

Passing to the limit as  $N \rightarrow \infty$ , we get (see the proof of Theorem 5.7 in [17])

$$|P_{t_2} f(x) - P_{t_1} f(x)| \leq C_3 e^{-\omega t_1} \|Df\|_0 (1 + |x|_H).$$

where  $C_3$  does not depend on  $t_1$ . This estimate shows that  $P_t f(x)$  converges to some constant exponentially fast as  $t \rightarrow +\infty$ . By ergodicity of the system, this constant must be  $\mu(f)$ .

Let us consider now  $\alpha \in (0, 1]$ . For any  $f \in C_b^1(H, \mathbb{R})$ ,  $s \in (0, 1]$ , define  $[f]_s$  by

$$[f]_s := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^s}.$$

We have

$$(5.2) \quad [f]_s \leq 2^{1-s} \sup_{x \neq y} \frac{|f(x) - f(y)|^s}{|x - y|^s} \|f\|_0^{1-s} \leq 2^{1-s} \|f\|_0^{1-s} \|Df\|_0^s.$$

Now we choose  $p \in (0, \alpha)$ . Using that  $\mathbb{E}|Z_A(t)|_H^p < \infty$  (see [17, Theorem 4.4]) and (5.2) with  $s = p$  we get, arguing as before,

$$\begin{aligned} |P_{t_2}^N f(x^N) - P_{t_1}^N f(x^N)| &= |\mathbb{E}[P_{t_1}^N f(X^N(s, x^N)) - P_{t_1}^N f(x^N)]| \\ &\leq 2^{1-p} \|DP_{t_1}^N f\|_0^p \|f\|_0^{1-p} \mathbb{E}|X^N(s, x^N) - x|_H^p \\ &\leq 2^{1-p} C^p e^{-\omega p t_1} \|f\|_{C_b^1} \mathbb{E}|X^N(s, x^N) - x^N|_H^p. \end{aligned}$$

Since

$$\mathbb{E}|Z_A(t)|_H^p \leq \tilde{c}_p \left( \sum_{n \geq 1} |\beta_n|^\alpha \frac{(1 - e^{-\alpha \gamma_n t})}{\alpha \gamma_n} \right)^{p/\alpha} \leq \tilde{c}_p \left( \sum_{n \geq 1} |\beta_n|^\alpha \frac{1}{\alpha \gamma_n} \right)^{p/\alpha} < +\infty, \quad t \geq 0,$$

it follows that

$$\mathbb{E}|X^N(s, x) - x^N|_H^p \leq C_4(1 + |x^N|_H)^p,$$

where  $C_4$  does not depend on  $N$ ,  $s \geq 0$  and  $x \in H$ . Finally we have

$$|P_{t_2} f(x) - P_{t_1} f(x)| \leq C_5 e^{-\omega p t_1} \|f\|_{C_b^1} (1 + |x|_H)^p,$$

where  $C_5$  does not depend on  $t_1$ . Arguing as before we complete the proof.

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